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# A STUDY FOR THE MATHEMATIC MODELING OF 2D IRREGULAR SHAPES FOR FOOTWEAR CAD SYSTEM

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***Abstract:*** *For using a specialized footwear CAD system it's imperative to know the analytical expression of the outlines of the footwear patterns. This brings us to the field of mathematical modeling. Mathematic modeling is based on the equation of the function defining the outline of the model contour. Shapes, contours cannot be identified, in designing, by simple function of the form y=f(x), because most of them have irregular forms, with many concavities and convexities, wich explains why their form is intrisically dependent on the coordinates system. For example, if we want to plot a curve, it is absolutely necesary that we choose the right set of contour points in a system of coordinates, but the important factor in determining the form of the object is the relation between these points, not that between the points and the randomly chosen coordinates system.Furthe more, the contour forms may have vertical tangents. If the shape were represented by a function y=f(x), the vertical tangents would be an incovenient in designing, which might be avoided by an aproximation of analytic function (e.g. of polynomials) For all these reasons, the dominant representation of shapes in CAD is not possible a function y=f(x) but a set of function wich can be obtained on various partions.*
*This paper presents a study regarding the interpolation of the footwear components and outlines contours and the graphic visualization, using the following methods: Lagrange, B-Spline, Bezier.*

***Key words:*** *footwear pattern, curves, interpolation, function*

## 1. MATHEMATIC MODELLING OF COMPONENTS. REQUIEREMENTS

Geometrical shapes commonly used in **Computer Aided Design** (**CAD**) systems can be defined by several points obtained through the digitizing process. The co-ordinates of the points situated between two nodes can be approximated through both analytic and graphic methods, with interpolation curves. Thus, the analytic expression of the curve that approximates the points will be a interpolation function. The graphical form will be represented by a curve that crosses all the co-ordinates of the digitized points, without bringing any mutations of the initial curve.

The analytic functions, approximating the contour to be designed, may be obtained by extrapolation if and only if there has been made a numeric coding of the geometric body, wich should furnish all necesary data.

As, however, many contours have irregular forms, the mathematic models are aproximative. Taking into account the advantage of computation technigue for these last years, we can asseses that highly performant programmes lead to approximations with minimum of errors.

Mention has to be of the processing centers with numerical command existent in highly developed countries, where, on the basis the cooditates of a set of points, the model to be designed in physically made up with the desired accuracy.

On a local scale, although the description and modelling of bodies, wich is an initial stage of data input, is slow ( form keyboard) or very expensive ( with specialized equipment) all efforts will be warranted by the spectacular results obtained.

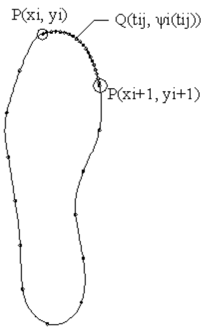


***Fig. 1****: The points Q between $P(x_i, x_{i+1})$ and $P(x_{i+1},y_{i+1})$ are defining for the coordinates $(t_{i,j}, \psi_j(t_{i,j}))$ with $t_{i,j} \in (x_i, x_{i+1})$*

Therefore, being given a finite number of points describing a curve, an interpolation problem is to be solved.

**1.1 Interpolation methods in Mathematic Modelling.**

Interpolation is a special case of a more general, approximating problem [1].A function f(t) will be approximated by interpolation as a finite sum: of a simple function $\psi_i$ , so as to satisfy a certain set of constraints for function g(t). As the constants $c_1,c_2,...,c_n$ have to be determined, it is necessary that the restrictive conditions for function g(t) should be specified.

Usually these conditions are imposed on the function g(t) so as to be a goot aproximation for function f(t).

If function g(t) is continuous over the approximating interval [a,b] then the function $\psi_1(t)$, $\psi_2(t)$, ...$\psi_n(t)$, continuous on [a,b], will be chosen. In the approximating theory, the constraints imposed on function g(t) will very often be the following:

- interpolation constraints:

$g(t_i)=f(t_i)$ for i€[1,n] (1)

in other words, the functions curves will have the same number of values within a finite number of points;

- mixture of interpolation and constraints:

a) $g(t_i) = f(t_i)$ for i€[1,k], k<n,
b) $g'(t_1) = f'(t_1)$ and $g'(t_k) = f'(t_k)$, (2)
c) g(t) - twice differential function;

- variational constraints:

$\|f-g\| = \min\{\|f-h\| / h$ € distance$(\psi_1, \psi_2, ...\psi_n)\}$, (3)

i. e. constants $c_1,c_2,...,c_n$ should be chosen in such a way as to allow that the minimum of a possible functions $\|f - h\|$ should be obtained of the set of all possible linear combinations:

$h = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + ...c_n\psi_n$ (4)

In order to accurately model it is necessary to pay special attention to certain qualities wich are characteristic to a set of curves under specific conditions.Some call this quality a shape characteristic.

A whole series of interpolating methods are dealt with in the literature [2]. In this paper we will present methods and results for following interplolation methods:

- Lagrange interpolation;
- Bezier interpolation;
- B-Spline interpolation.

## 2. LAGRANGE INTERPOLATION

The Lagrange interpolation is a classic method of interpolation. The Lagrange polynomials are among the simplest interpolating polynomials [3], [4]. [5], [6].

Being given n+1 discrets points:

$x_0<x_1<x_2<...<x_n$ (5)

and a second set of real numbers:

$y_1, y_2, y_3, \ldots, y_n$ (6)

there may be defined the Lagrange polynomial of nth power, associated to the two sets of number $\{x_i, y_i\}$ as being the polynomial P(x) wich solves the interpolation problem:

$P(x_i) = y_i, \quad i \in [0,n]$ (7)

where $x_i$ are called nodes and the $y_i$ constraints of interpolation function.

The advantage of this way of interpolating is that these polynomials are single. If a function is defined in $x_0, x_1, x_2, \ldots, x_n$ and the corresponding $y_0, y_1, y_2, \ldots y_n$ values are known, then:

$f(x_i) = y_i \quad i \in [0,n],$ (8)

and by Lagrange interpolation of the nth power of the function f(x) in nodes $x_i$ there results polynomial:

$$P_n(x) = \sum_{i=1}^{n} f(x_i) L_{i,n}(x) \quad (9)$$

with:

$$L_{i,n} = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x-x_n)} \quad (10)$$

or expresising the classic Lagrange polynomial:

$P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1}$ (11)

It results from the theoretical considerations mentioned above that mathematic modelling by Lagrange method implies solving a system of n+1 equations with n+1 unknows of the form:

$$\begin{aligned} y_0 &= a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + \ldots + a_nx_0^n \\ y_1 &= a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + \ldots + a_nx_1^n \\ y_2 &= a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 + \ldots + a_nx_2^n \\ \\ y_n &= a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + \ldots + a_nx_n^n \end{aligned} \quad (12)$$

where $(x_i, y_i)$ with $i \in [0,n]$ are the points describing the forme to be indentified.

Therefore, the problem consists in determining the polynomial coeficients $a_0, a_1, a_2, \ldots, a_n$ by solving a system of n + 1 equations with n +1 unknowns.

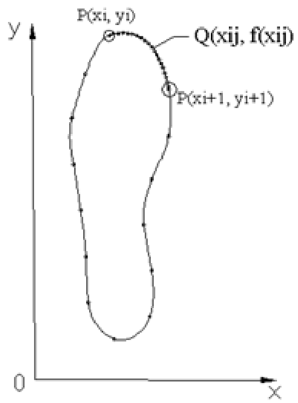


***Fig. 2**: The points Q between $P(x_i, y_i)$ and $P(x_{i+1}, y_{i+1})$ are defining for the coordinates $(x_{i,j}, f(t_{i,j}))$ where f is the Lagrange polynom*

### 2.1 Experimental estimations

For mathematical modelling of a contour by Lagrange interpolation will considered a contours of a footwear part to wich a system of XoY coordinates was attached. The contour was divided in several partions of convenient size, as shown in Fig. 2, after wich the coordinates $(x_i, y_i)$ of the chosen points, necessary for Lagrange interpolation, were selected.

NOTE

Mention is made to the fact that an accurated modelling of a number of points the nth power of the polynomial does not have to be too big, because is ocurring errors and oscillations in displaying the polynomial. For that, we will utilize the parametrically represented.

## 3. THE PARAMETRICALLY REPRESENTED

The shapes of footwear parts cannot be precisely described by a simples function y=f(x). In CAD the shapes are parametricaly represented using a set of functions [4]. [5], [6]:

$x = x(t); \qquad y = y(t).$ (13)

In order to do that, we will attach two supplementary systems of co-ordinates – *tOx* and *tOy* – to the present one – *xOy*. We select the variation domain of parameter t. Then, we solve the two independent problems of theoretical interpolation for the two variables, x and y using many methods.

The parametric equations used in interpolation are actually polynomial equations, usually bicubic, described as:

$g(t)=at^3+bt^2+ct+d$ (14)

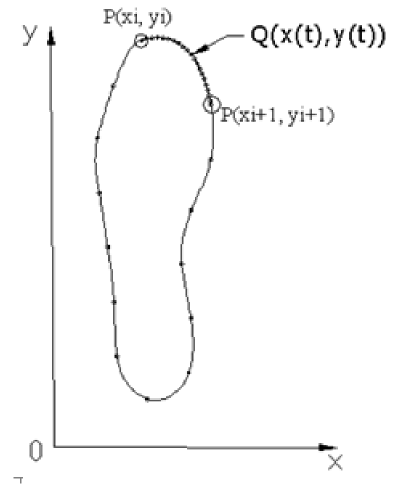


***Fig. 3:*** *The points Q between $P(x_i, y_i)$ and $P(x_{i+1},y_{i+1})$ are defining for the coordinates $(x(t),y(t))$ two interpolating polynoms*

Thus, we will the set of points $(x_0, y_0), (x_1, y_1)\ldots (x_{n-1}, y_{n-1})$ by the aid of two variables $x(t)$ and $y(t)$, defined by two parametric interpolation equations:

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y \qquad (15)$$

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### 3.1. Defining Bezier interpolating polynoms

This will allow the determination of the four coefficients from the following restrictive conditions:

1. The value of the polynom in the nodes must be the same with its numeric value:

$x(t_i)=x_i,\ y(t_i)=y_i$ (16)

where i=0… n and $x_i$ and $y_i$ are nodes: $(x_0, y_0), (x_1, y_1)\ldots (x_n, y_n)$

2. The vector that is tangent to the curve in a node must be the same as the vector of the initial curve. If the components of the vector tangent to the interpolating curve are:

$$x'(t) = 3a_x t^2 + 2b_x t + c_x$$
$$y'(t) = 3a_y t^2 + 2b_y t + c_y \qquad (17)$$

then the restrictive conditions for the limit will be:

$x'(t_i)=l_i$ and $y'(t_i)=m_i$ (18)

where $l_i$ and $m_i$ are the values of the incline of the tangents in the nodes, as determined in the tOx and tOy co-ordinates systems.

#### 3.1.1.Theoretically solving the interpolation Bézier curves.

Without affecting the general area of an interpolation problem, the interpolating Bézier polynoms derive from the following conditions [4],-5]:

**C1**. We take the interval [0,1] as a variation domain for parameter t. In this case, the conditions for the limit will be:

$x(0)=d_x \qquad y(0)=d_y$

$x(1)=a_x+b_x+c_x+d_x \qquad y(1)=a_y+b_y+c_y+d_y$ (19)

**C2**. The directions of the tangents in the nodes will be defined as the inclination of the tangent led in every node. For example, in the $(x_i,y_i)$ node, they will be calculated with the following relations:

$l_i=m(x_i-x_{c1}) \qquad m_i=m(y_i-y_{c1})$ (20)

$l_{i+1}=m(x_{i+1}-x_{c2}) \qquad m_{i+1}=m(y_{i+1}-y_{c2})$

where:

- *m*, known as a **shape factor** in the literature, usually takes the value 3,
- $x_i,y_i$ and $x_{i+1},y_{i+1}$ are **the co-ordinates of the nodes** (the extreme point of the curves),
- $x_{c1},y_{c1}$ and $x_{c2},y_{c2}$ are the co-ordinates of the two points that belong to the tangents to the Bézier curve and they are called **points of control**.

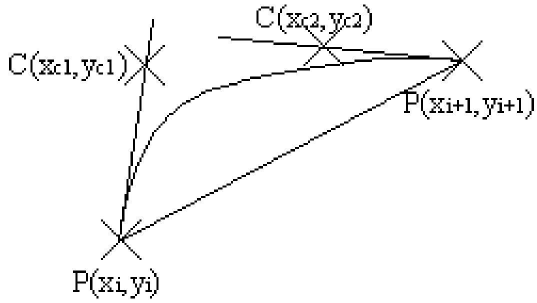


***Fig. 4** The Bezier polynom is defined for two nodes and two points of of* control

This relations and of the two conditions lead to a system of 8 equations with 8 unknown values, with the following solutions:

$d_x=x_i \qquad d_y=y_i$

$c_x=3(x_{c1}-x_i) \qquad c_y=3(y_{c1}-x_i)$

$b_x=3(x_{i+1}-x_i)-c_x \qquad b_y=3(y_{i+1}-y_i)-c_y$

$a_x=x_{c2}-x_i-c_x-b_x \qquad a_y=y_{c2}-y_i-c_y-b_y$ (21)

This represents the mathematical expression of the coefficients of the bicubic polynomial Bezier functions.

### 3.1.2. Discussions

If we analyse the two conditions, the conclusions will be as follows:

1. A Bézier curve is defined by four points:

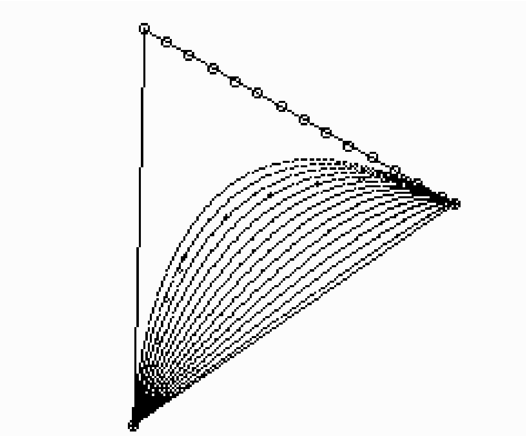

***Fig. 5**: Between two nodes we can define many Bézier curves*

• two fixed points on the Bézier curve (nodes), that are fixed;

- • two other intermediate points, that belong not to the curve, but to its tangents. The two points are called **control points** and are positioned on the tangents of the curve led in the nodes( $P(x_i,y_i)$, $P(x_{i+1},y_{i+1})$, *see picture 4*).

.2. Between two nodes we can define many Bézier curves, related to the position of the control point on the tangent *(see picture 5)*. This makes it possible, in a CAD work session, to draw several Bézier curves and choose the convenient one – the one that approximates a set of points between the two nodes with the highest precision.

### 3.2 Defining B-Spline interpolating polynoms

Let's take into consideration a network of points called $t_0<t_1<.....t_n$ whose nodes have known values given by $\{f_i\}$, $i\in[0,n]\rightarrow R$, where $\{f_i\}$ is an interpolating B-spline function that fulfills the following conditions:

- g(t) is continuous on the $(t_0,t_n)$ interval, together with its first and second rank derivates.
- on each $[t_{i-1},t_{i+2}]$ interval, the function is a third degree polynom, $i\in[1,n-1]$

- in the nodes of the *{$t_i$}, $i \in [0,n]$* network, the following conditions are fulfilled:
  $g(t_i)=f_i$, $i \in [0,n]$; $g'$ fulfills the conditions of the limit: $g'(t_{i-1})=f'(t_{i-1})$
  $g'(t_{i+2})=f'(t_{i+2})$ $g''$ fulfills the conditions of the limit: $g''(t_i)=f''(t_i)$

**3.2.1 The mathematical expression of the coefficients of the bicubic polynomial B-Spline functions**

The B-spline functions are thirdy degree polynoms on portions defined by the pointes $t_{i-1},t_i,t_{i+1},t_{i+2}$.

By applying the same methodology as presented in the previous paragraph, the third degree polynom representing the interpolating B-spline function looks like:

$x(t)=a_x t^3+b_x t^2+c_x t+d_x$

$y(t)= a_y t^3+b_y t^2+c_y t+d_y$

P where:

$$a_x=(-x_{i-1}+3x_i - 3x_{i+1} + x_{i+2})/6 \qquad a_y=(-y_{i-1}+3y_i - 3y_{i+1} + y_{i+2})/6$$
$$b_x= (3x_{i-1}-6x_i+3x_{i+1})/6 \qquad b_y= (3y_{i-1}-6y_i+3y_{i+1})/6$$
$$c_x = (-3x_{i-1} + 3x_{i+1})/6 \qquad c_y = (-3y_{i-1} + 3y_{i+1})/6 \qquad (11)$$
$$d_x = (x_{i-1} +4x_i + x_{i+1})/6 \qquad d_y = (y_{i-1} +4y_i + y_{i+1})/6$$

for $i \in [1,n-1]$.

The expressions lead to the conclusion that the interpolating B-spline function can be determined only by four successive coordinates of the shape.

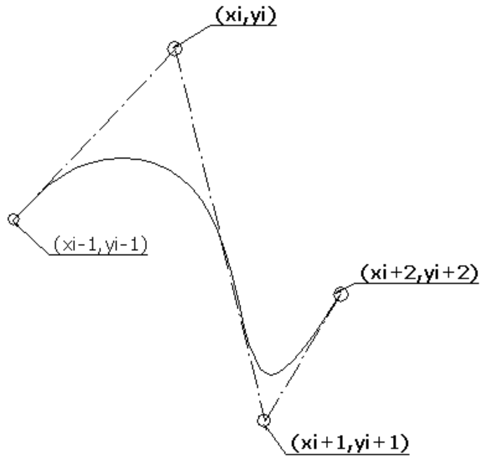


***Fig. 6**: Four successive coordinates determined a curves B-Spline*

## 4. CONCLUSIONS

After those presented, it can be noticed that a point belonging to a B-spline polynom depends on the coordinates of its neighboring points, but it doesn't depend at all of all the points defining the shape. In the words of specialists, this means that "a B-spline approximation has a local scheme".

The characteristic of being able to decrease the variation and the local character of the approximation when using a B-spline function can be explained by looking at the geometrical behavior of the B-spline curves.

- a point on such a curve is a convex combination of K neighboring nodes of the polygon and thus the curve can very well "fit" with the polygon.
- if one node of the polygon is moved, the curve will only be modified locally, in the neighborhood of the node. This characteristic is essential in design, because it allows the creator to modify a curve he is not satisfied with, without having to re-make the whole outline.

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